MATH 5061 Lecture 3 (Jan 27)

[Problem Set 2 is posted, due on Feb 10.]

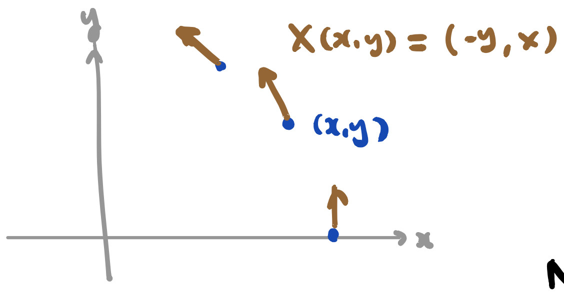
Last time ...... $f: M \to N$ embedding, TM, vector bundle (on S^n)

vector field X on M, as a section of TM.

§ Vector Fields as "derivations"

$X \in \Gamma(TM)$ locally in coord $X(x_1, \dots, x_m) = \sum_{i=1}^{m} X^i(x_1, \dots, x_m) \frac{\partial}{\partial x^i}$

E.g.) In $\mathbb{R}^2$,



write: $X(x,y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$.

Let $f(x,y) = xy$.

$$X(f) = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} = -y^2 + x^2$$

Note: $X : f \mapsto X(f)$

IDEA: X acts on smooth functions $C^\infty(M)$ by directional derivative

Notation: $C^\infty(M) := \{ f : M \to \mathbb{R} \text{ smooth} \}$.

$\mathrm{Diff}(M) := \{ \varphi : M \to M \text{ diffeo.} \}$.

Given $X \in \Gamma(TM)$, $f \in C^\infty(M)$, $p \in M$,

$$X(f)(p) := \sum_{i=1}^{m} X^i(0) \left. \frac{\partial f}{\partial x^i} \right|_0$$

for any local coord. $x^1, \dots, x^m$ s.t. $p = 0$.

Consider all points $p \in M$,

$$\Gamma(TM) \ni X : \begin{array}{ccc} C^\infty(M) & \longrightarrow & C^\infty(M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & X(f) \end{array}$$

Prop: The map above is a derivation, i.e. $\forall\, a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$,

(1) "Linearity": $X(af + bg) = a\,X(f) + b\,X(g)$

(2) "Liebniz Rule": $X(fg) = g \cdot X(f) + f \cdot X(g)$

FACT: $\left\{ \begin{array}{c} \text{vector fields} \\ \text{on } M \end{array} \right\} \underset{\text{corr.}}{\overset{1-1}{\longleftrightarrow}} \left\{ \begin{array}{c} \text{derivations} \\ \text{on } M \end{array} \right\}$

Def^{n}: (Lie bracket) Let $X, Y \in \Gamma(TM)$.

$$[X, Y] := XY - YX \in \Gamma(TM).$$

i.e. $[X,Y](f) := X(Y(f)) - Y(X(f))$

Properties of $[\cdot,\cdot]$

(i) $[X, Y] = -[Y, X]$

(ii) $[\cdot,\cdot]$ is $\mathbb{R}$-linear in each slot

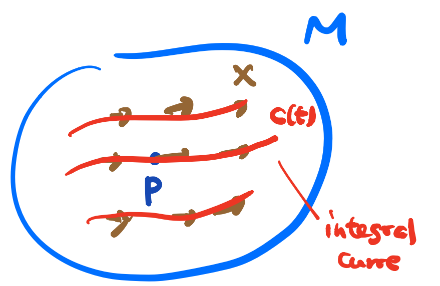
(iii) (Jacobi identity) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

Caution: $[\cdot,\cdot]$ is defined only using the smooth structure on M.

§ Flow and integral curves of vector fields

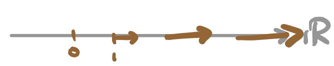
Let $X \in \Gamma(TM)$. Consider the following I.V.P.

$$\begin{cases} c_p'(t) = X(c_p(t)) & \forall\, t \in I \\ c_p(0) = p \end{cases}$$



O.D.E. $\Rightarrow \exists$ unique sol^{n} $c_p(t) : I_p \to M$ that depends smoothly on the initial data $c(0) = p$

Eg. $X = x^2 \frac{\partial}{\partial x}$.



Thm: If $X \in \Gamma(TM)$ is compactly supported, then the maps

$$\begin{array}{cccc} \phi_t : & M & \longrightarrow & M \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & p & \longmapsto & C_p(t) \end{array}$$

is a diffeo. for each $t \in \mathbb{R}$.

Moreover, $\phi_t \circ \phi_s = \phi_{t+s} \quad \forall t, s \in \mathbb{R}$.

i.e. $\{\phi_t\}_{t \in \mathbb{R}} \subseteq \mathrm{Diff}(M)$ forms a 1-parameter group

called the flow generated by X.

Remarks: · If X not cptly supported, we can still define maps locally.

· Any $\phi \in \mathrm{Diff}(M)$ induces a pushforward map

$$\phi_* : \Gamma(TM) \longrightarrow \Gamma(TM)$$

by the differential $d\phi_p : T_pM \to T_{\phi(p)}M$ at each $p \in M$.

Thm: Let $X, Y \in \Gamma(TM)$, cptly supported.

Suppose $\{\phi_t\}_{t \in \mathbb{R}}$ is the flow generated by Y.

Then.
$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} (\phi_t)_* X \quad (=: -\mathcal{L}_Y X)$$

§ Tensors

Recall: (Linear Algebra)

dim: $\dim V$ $\quad \dim V + \dim W$ $\quad \dim V \times \dim W$

V, W vector space ($/\mathbb{R}$) $\rightsquigarrow$ V^*, $V \oplus W$, $V \otimes W$

V^*: dual space; $V \oplus W$: direct sum; $V \otimes W$: tensor product.

Tensor Product:

$\dim(V \otimes W) = \dim V \cdot \dim W$.

$$V \otimes W := \left\{ \sum_{i=1}^{k} a_i (v_i \otimes w_i) \;\middle|\; a_i \in \mathbb{R},\ v_i \in V,\ w_i \in W \right\}$$

s.t.
$$\left.\begin{aligned} (a_1 v_1 + a_2 v_2) \otimes w &= a_1 (v_1 \otimes w) + a_2 (v_2 \otimes w) \\ v \otimes (b_1 w_1 + b_2 w_2) &= b_1 (v \otimes w_1) + b_2 (v \otimes w_2) \end{aligned}\right\} \text{"bilinearity" } \cdot \otimes \cdot$$

Equivalently, we view:

$$V^* \otimes W^* \cong \{ \phi : V \times W \longrightarrow \mathbb{R} \text{ "bilinear"} \}$$

i.e. $\phi(\cdot, w) : V \to \mathbb{R}$ linear for each fixed $w \in W$

$\phi(v, \cdot) : W \to \mathbb{R}$ linear for each fixed $v \in V$.

Recall: $\exists$ natural / canonical pairing

$$\begin{array}{ccc} V \times V^* & \longrightarrow & \mathbb{R} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (v, v^*) & \longmapsto & v^*(v) \end{array}$$

We have for any $v^* \in V^*$, $w^* \in W^*$,

$$V^* \otimes W^* \ni (v^* \otimes w^*)(v, w) := v^*(v) \cdot w^*(w)$$

We can define tensor product of linear maps as follows:

Given linear maps $T : V \to \tilde{V}$, $S : W \to \tilde{W}$.

$$\begin{array}{cccc} T \otimes S : & V \otimes W & \longrightarrow & \tilde{V} \otimes \tilde{W} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & v \otimes w & \longmapsto & T(v) \otimes S(w) \end{array}$$

Ex: Well-defined?

Moral: Any "canonical" (i.e. indep. of choice of basis) constructions for vector spaces can be done fiberwise on vector bundles.

Applying to the tangent bundle TM

$$TM := \coprod_{p\in M} T_pM \xrightarrow{\text{dual}} T^*M := \coprod_{p\in M} T_p^*M \quad \text{cotangent bundle}$$

$$T^{r}_{s}M := \coprod_{p\in M} (\underbrace{T_pM \otimes \cdots \otimes T_pM}_{r\text{-times}}) \otimes (\underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{s\text{-times}})$$

contravariant

covariant

(r,s)-tensor bundle over M

E.g.) $T^1_0M = TM$; $T^0_1M = T^*M$

Def$^{\underline{n}}$: $\Gamma(T^r_sM) := \{(r,s)\text{-tensors on } M\}$ "$C^\infty(M)$-module"

Some algebraic tensor operations

① tensor product $\otimes$

② "Contraction": $C_{i,j} : V^{\otimes p} \otimes V^{*\otimes q} \longrightarrow V^{\otimes(p-1)} \otimes V^{*\otimes(q-1)}$
(w.r.t. i, j)

$$C_{i,j}(v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^*)$$

$$= \underbrace{v_j^*(v_i)}_{\in\mathbb{R}} (v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_p) \otimes (v_1^* \otimes \cdots \otimes \widehat{v_j^*} \otimes \cdots \otimes v_q)$$

E.g.) $C_{1,1} : V \otimes V^* \longrightarrow \mathbb{R}$; $C_{1,1}(v \otimes v^*) = v^*(v)$

Note: This is just the "trace" on $\mathrm{End}(V) \cong V \otimes V^*$

Ex: Check this!

i.e. $(v \otimes v^*)(w) = v^*(w) \cdot v$

③ "<u>Interior Product</u>" (w.r.t $v \in V$)

Given $\alpha \in (V^*)^{\otimes q}$, i.e. $\alpha : \overbrace{V \times \cdots \times V}^{q\text{-times}} \to \mathbb{R}$ multilinear.

define $(\iota_v \alpha) \in (V^*)^{\otimes (q-1)}$ as

$$(\iota_v \alpha)(v_1, \ldots, v_{q-1}) := \alpha(v, v_1, \ldots, v_{q-1})$$

Pullback of tensors

Given a diffeo. $\phi : M \to N$, we can pullback (p,q)-tensors on N to obtain (p,q)-tensors on M as follow:

$$\begin{tikzcd} \phi^*\alpha & \alpha \arrow[l, bend right, "\phi^*"'] \end{tikzcd} \quad (p,q)\text{-tensor}$$

$$\begin{tikzcd} M \arrow[r, "\phi", "\text{diffeo.}"'] & N \arrow[l, bend left, "\phi^{-1}"] \end{tikzcd}$$

$$\phi^* : \mathcal{T}(T^p_q N) \to \mathcal{T}(T^p_q M)$$

s.t. • $\phi^*(X) = (\phi^{-1})_* X \quad \forall X \in \mathcal{T}(TN)$

vector fields

• $(\phi^*\alpha)(X)(x) = \alpha_{\phi(x)}(d\phi_x(X)) \quad \forall \alpha \in \mathcal{T}(T^0_1 N)$

1-forms

$\forall X \in \mathcal{T}(TM),\ x \in M$

• $\phi^*(\alpha \otimes \beta) = \phi^*\alpha \otimes \phi^*\beta \quad \forall$ tensors α, β of any type.

<u>Remarks</u>: (i) $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ for $\phi, \psi \in \mathcal{D}iff$.

(ii) ϕ^* commutes with any contraction.

§ Lie derivative

Given $X \in \mathcal{T}(TM)$, we can define the Lie derivative (w.r.t X)

flow $\{\phi_t\}_t$

$$\mathcal{L}_X : \mathcal{T}(T^p_q M) \longrightarrow \mathcal{T}(T^p_q M)$$

by

$$\mathcal{L}_X \alpha := \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* \alpha)$$

Properties of $\mathcal{L}_X$

(a) $\mathcal{L}_X f = X(f) = df(X)$, $\forall f \in C^\infty(M)$

(b) $\mathcal{L}_X Y = [X, Y] \quad \forall Y \in \Gamma(TM)$

(c) $\mathcal{L}_X(\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes (\mathcal{L}_X \beta)$ $\forall$ tensors α, β.

(d) $\mathcal{L}_X \circ C = C \circ \mathcal{L}_X$ $\forall$ contraction C

FACT: These 4 properties uniquely characterize $\mathcal{L}_X$.

Reason: Suppose $\exists$ linear map

$$P_X : \Gamma(T^p_q M) \to \Gamma(T^p_q M)$$

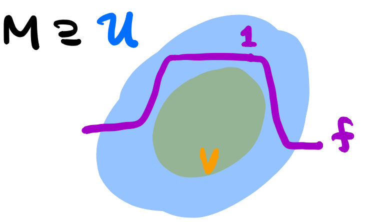
satisfying (a) - (d) above. Claim: $P_X = \mathcal{L}_X$.

First, we show P_X is a "local" operator:

i.e. Suppose $\alpha, \beta \in \Gamma(T^p_q M)$ s.t. $\alpha|_{\mathcal{U}} \equiv \beta|_{\mathcal{U}}$ on some open $\mathcal{U} \subseteq M$.

Then, $(P_X \alpha)|_{\mathcal{U}} \equiv (P_X \beta)|_{\mathcal{U}}$

Why? Choose another open $V \subset\subset \mathcal{U}$, ie $\bar{V} \subset \mathcal{U}$.



Fix $f \in C^\infty(M)$ cutoff fcn st

$$\begin{cases} f \equiv 1 & \text{on } V \\ f \equiv 0 & \text{outside } \mathcal{U}. \end{cases}$$

Now, $\alpha|_{\mathcal{U}} \equiv \beta|_{\mathcal{U}} \Rightarrow f\alpha = f\beta$ on M

(c) $\Rightarrow \underbrace{(P_X f)}_{(a)\ X(f)} \alpha + f(P_X \alpha) = \underbrace{(P_X f)}_{X(f)} \beta + f(P_X \beta)$ on M

$\Rightarrow \quad f(P_X \alpha) = f(P_X \beta)$ on $\mathcal{U}$

$\Rightarrow \quad P_X \alpha = P_X \beta$ on V $\Rightarrow$ also on $\mathcal{U}$ since V arbitrary.

<u>Application:</u> $L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$

Q: What is a tensor "really"?

$$
\begin{array}{ccc}
(1,0)\text{-tensor} & \longleftrightarrow & \text{vector fields} \\
\updownarrow \text{dual} & & \updownarrow \text{dual} \\
(0,1)\text{-tensor} & \longleftrightarrow & 1\text{-form}
\end{array}
$$

- What about $(0,2)$-tensors? $C^\infty(M)$-module

$$(0,2)\text{-tensors} \longleftrightarrow \text{map } \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$$

bilinear over $C^\infty(M)$

<u>Why?</u> "$\Rightarrow$" Given $(0,2)$-tensor $\alpha \in \Gamma(T_2^0 M)$, we define

$$\alpha : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$$

s.t. $\alpha(X,Y)(p) = \alpha_p(X_p, Y_p) \quad \forall p \in M$

$\alpha_p \in T_p^*M \otimes T_p^*M$

<u>Note:</u> $\alpha(fX, Y) = f\,\alpha(X,Y) = \alpha(X, fY)$, $\forall f \in C^\infty(M)$

"$\Leftarrow$" Given a map

$$\Psi : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M), \quad C^\infty(M)\text{-bilinear}$$

At each $p \in M$, we define a bilinear map (over $\mathbb{R}$)

$$\alpha_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

as follows, let $X_p, Y_p \in T_pM$, extending to $X, Y \in \Gamma(TM)$ (non-unique)

$$\Psi(X,Y)(p) =: \alpha_p(X_p, Y_p)$$

Ex: uses bilinearity / $C^\infty(M)$ $\Rightarrow$ well-defined $\because$ indep of the extensions X, Y

Application: $[\cdot,\cdot] : \mathcal{T}(TM) \times \mathcal{T}(TM) \to \mathcal{T}(TM)$ is **NOT** a tensor

Digression: $\mathrm{Hom}(V,W) \cong V^* \otimes W$; $(V \otimes W)^* \cong V^* \otimes W^*$

$$\underset{C^\infty(M)\text{-linear}}{\Psi : \mathcal{T}(T^p_q M) \to \mathcal{T}(T^r_s M)} \iff \Psi \in \mathcal{T}\big(\underbrace{(T^p_q M)^* \otimes T^r_s M}_{T^{q+r}_{p+s} M}\big)$$

Since $[fX, gY] = \underbrace{(f\cdot(Xg))Y - (g\cdot(Yf))X}_{\text{NOT } C^\infty(M)\text{-biline}} + fg\,[X,Y]$

$f, g \in C^\infty(M)$